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A Note on Semilinear Abstract Functional Differential and Integrodifferential Equations with Infinite Delay

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Abstract—In this note, we obtain some new existence and uniqueness theorems for mild solutions of the Cauchy problems for semilinear abstract functional differential and integrodifferential equations with infinite delay.

Keywords—Mild solutions, Families of operators, Semilinear abstract functional differential equations, Semilinear abstract functional integrodifferential equations, Infinite delay.

Write $R^- = (-\infty, 0]$ and $x_t(\theta) = x(t+\theta)$ $(\theta \in R^-)$. Let X be a Banach space and $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ be a Banach space consisting of functions from R^- into X satisfying the following basic axioms.

(H1) For any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0 + a] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in [t_0, t_0 + a]$.

(H2) $\|\phi(0)\| \le K\|\phi\|_{\mathcal{P}}$ for all $\phi \in \mathcal{P}$ and a constant K.

(H3) There exist nonnegative, measurable, and locally bounded functions $K(t)$ and $M(t)$ of $t \ge 0$ such that

$$\|x_t\|_{\mathcal{P}} \le K(t - t_0) \sup_{s \in [t_0, t]} \|x(s)\| + M(t - t_0)\|x_{t_0}\|_{\mathcal{P}}$$

for $t \in [t_0, t_0 + a]$ and x as in (H1).

For the examples of the spaces satisfying (H1)–(H3) please see, e.g., [1–4] and see, e.g., [5,6] for the case of $X = R^n$. As a continuation of the works [1–4], we investigate the existence and

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uniqueness of mild solutions of the Cauchy problem for abstract semilinear functional differential equations with infinite delay

$$u'(t) = Au(t) + f(t, u(t), u_t) \quad (0 \le t \le T), \qquad u_0 = \phi, \tag{1}$$

and the Cauchy problem for abstract semilinear integrodifferential equations with infinite delay

$$\begin{aligned} u'(t) &= A\left[u(t) + \int_0^t F(t-s)u(s)\,ds\right] + f(t, u(t), u_t), \qquad 0 \le t \le T, \\ u_0 &= \phi, \end{aligned} \tag{2}$$

where $T > 0$, $\phi(\theta) \in \mathcal{P}$, A is a closed linear operator in X, $\{F(t)\}_{0 \le t \le T} \subset \mathbf{L}(X)$ (the space of bounded linear operators from X to X), and $f \in C([0,T] \times X \times \mathcal{P}, X)$. It is known that the equations with delay (i.e., with some of the past states of the systems), compared with those without delay, are more realistic to describe many phenomena in nature, and they have a very strong application background. See [1–6,8,9, VI.6; 10, Sect. 15.18; 11,12] and references therein for more comments. Also, the point that the space X may be infinite-dimensional (not only finite-dimensional) would allow a wide applicability of our theorems (cf., e.g., [1–4,9]).

THEOREM 1. *Let $0 \le \sigma < T$, $f \in C([\sigma,T] \times X \times \mathcal{P}, X)$ and satisfy that for every $\sigma \le \tau < T$ and $r > 0$ there exists a constant $H(\tau, r)$ such that for any $t \in [\sigma, \tau]$,*

$$\|f(t,u,\phi) - f(t,v,\psi)\| \le H(\tau,r)\left(\|u-v\| + \|\phi-\psi\|_{\mathcal{P}}\right), \qquad \|u\|, \|v\|, \|\phi\|_{\mathcal{P}}, \|\psi\|_{\mathcal{P}} \le r. \tag{3}$$

Then for every $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,T],X)$ with $g(\sigma) = \phi(0)$, and strongly continuous family $\{E(t)\}_{t \ge 0} \subset \mathbf{L}(X)$, there exists a real number $\tau(\sigma, \phi, g, E(\cdot))$ such that the Cauchy problem for the integral equation

$$u(t) = g(t) + \int_\sigma^t E(t-s) f(s, u(s), u_s)\,ds \quad (\sigma \le t \le T), \qquad u_\sigma = \phi, \tag{4}$$

has a unique continuous solution $u(t)$ on $[\sigma, \tau(\sigma, \phi, g, E(\cdot)))$.

PROOF. For each $\tau > \sigma$, $b > 0$, $\phi \in \mathcal{P}$, set

$$\mathcal{P}^{[\sigma,\tau]} = \left\{u : (-\infty, \tau] \to X; u\,|_{[\sigma,\tau]} \in C([\sigma,\tau],X) \text{ and } u_\sigma \in \mathcal{P}\right\},$$

$$\mathcal{P}_{\phi,g}^{[\sigma,\tau]}(b) = \left\{u \in \mathcal{P}^{[\sigma,\tau]};\ \max_{t \in [\sigma,\tau]} \|u(t) - g(t)\| \le b, u_\sigma = \phi\right\}.$$

Then $\mathcal{P}^{[\sigma,\tau]}$ is a Banach space under the norm $\|u\|_{\mathcal{P}^{[\sigma,\tau]}} := \sup_{t\in[\sigma,\tau]} \|u(t)\| + \|u_\sigma\|_{\mathcal{P}}$, and $\mathcal{P}_{\phi,g}^{[\sigma,\tau]}(b)$ is closed and convex. Take a real number σ_0 such that $0 < \sigma_0 < T - \sigma$ and let

$$b_1 = \sup_{t \in [\sigma, \sigma+\sigma_0]} \{\|g(t)\|, K(t), M(t)\}, \qquad r = \max\{b + b_1, b_1(b + b_1 + \|\phi\|_{\mathcal{P}})\}.$$

Then the definition of $\mathcal{P}_{\phi,g}^{[\sigma,\sigma+\sigma_0]}(b)$ and (H3) imply that

$$\max_{t \in [\sigma,\sigma+\sigma_0]} \{\|u(t)\|, \|u_t\|_{\mathcal{P}}\} \le r, \qquad u \in \mathcal{P}_{\phi,g}^{[\sigma,\sigma+\sigma_0]}(b). \tag{5}$$

For every $u \in \mathcal{P}_{\phi,g}^{[\sigma,\tau]}(b)$ and $\sigma < \tau \le \sigma + \sigma_0$, define $\mathbf{F}u$ by

$$(\mathbf{F}u)(t) = \begin{cases} g(t) + \displaystyle\int_\sigma^t E(t-s) f(s, u(s), u_s)\,ds, & t \in [\sigma, \tau], \\ \phi(t - \sigma), & t \in (-\infty, \sigma]. \end{cases}$$

Then $\mathbf{F}u \in \mathcal{P}^{[\sigma,\tau]}$ by (H1). Moreover, by (3), (5), the strong continuity of $\{E(t)\}_{t\geq 0}$ and (H3), we obtain for each $u \in \mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b)$,

$$\begin{aligned}&\max_{t\in[\sigma,\tau]} \|(\mathbf{F}u)(t) - g(t)\| \\ &\quad\leq \max_{t\in[\sigma,\sigma+\sigma_0]} \|E(t)\| \max_{t\in[\sigma,\tau]} (\|[f(t,u(t),u_t) - f(t,0,0)]\| + \|f(t,0,0)\|)(\tau-\sigma) \\ &\quad\leq \text{const}\,(\tau-\sigma),\end{aligned}$$

and for all u, $v \in \mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b)$,

$$\begin{aligned}&\max_{t\in[\sigma,\tau]} \|(\mathbf{F}u)(t) - (\mathbf{F}v)(t)\| \\ &\quad\leq \max_{t\in[\sigma,\sigma+\sigma_0]} \|E(t)\| H(\sigma+\sigma_0, r)\left(\max_{t\in[\sigma,\tau]} \|u(t)-v(t)\| + \max_{t\in[\sigma,\tau]} \|u_t - v_t\|_{\mathcal{P}}\right)(\tau-\sigma) \\ &\quad\leq (\tau-\sigma)\ \text{const} \max_{t\in[\sigma,\tau]} \|u(t)-v(t)\|.\end{aligned}$$

Hence, for any given $\varepsilon < 1$, there is a real number $\sigma < \tau(\sigma,\phi,g,E(\cdot)) \leq \sigma + \sigma_0$ such that for every $\sigma < \tau < \tau(\sigma,\phi,g,E(\cdot))$,

$$\max_{t\in[\sigma,\tau]} \|(\mathbf{F}u)(t) - g(t)\| \leq b, \quad u \in \mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b),$$

$$\|(\mathbf{F}u)(t) - (\mathbf{F}v)(t)\|_{\mathcal{P}^{[\sigma,\tau]}} \leq \varepsilon \|u-v\|_{\mathcal{P}^{[\sigma,\tau]}}, \quad u, v \in \mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b).$$

Consequently, $\mathbf{F}$ has a unique fixed point in $\mathcal{P}^{[\sigma,\tau]}_{\phi,g}(b)$. This shows that for each $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,T],X)$ with $g(\sigma) = \phi(0)$, and strongly continuous family $\{E(t)\}_{t\geq 0} \subset \mathbf{L}(X)$, (4) has a continuous solution on $[\sigma, \tau(\sigma,\phi,g,E(\cdot)))$.

The uniqueness of the solution is implied by (H3), (3), and Gronwall-Bellman's inequality.

THEOREM 2. *Let $\sigma \geq 0$, $f \in C([\sigma,\infty) \times X \times \mathcal{P}, X)$ and satisfy (3) (for $T = \infty$). For any $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,\infty),X)$ with $g(\sigma) = \phi(0)$, and strongly continuous family $\{E(t)\}_{t\geq 0} \subset \mathbf{L}(X)$, put $T(\sigma,\phi,g,E(\cdot)) := \sup\{t > \sigma$; (4) has a unique continuous solution $u(\cdot)$ on $[\sigma,t)\}$. If $T(\sigma,\phi,g,E(\cdot)) < \infty$, then $\limsup_{t\uparrow T(\sigma,\phi,g,E(\cdot))} \|u(t)\| = \infty$.*

PROOF. Fix $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,\infty),X)$ with $g(\sigma) = \phi(0)$, and $\{E(t)\}_{t\geq 0} \subset \mathbf{L}(X)$ which is strongly continuous. Theorem 1 indicates that $T(\sigma,\phi,g,E(\cdot))$ is well defined. Let $u(t)$ be the corresponding solution to (4) on $[\sigma, T(\sigma,\phi,g,E(\cdot)))$, and suppose that $T(\sigma,\phi,g,E(\cdot)) < \infty$ and $\limsup_{t\uparrow T(\sigma,\phi,g,E(\cdot))} \|u(t)\| < \infty$. Then there exists a constant b_2 such that

$$\max\left\{\sup_{t\in[\sigma,T(\sigma,\phi,g,E(\cdot))+1]} \{\|g(t)\|, K(t), M(t)\}, \sup_{t\in[\sigma,T(\sigma,\phi,g,E(\cdot)))} \|u(t)\|\right\} \leq b_2. \tag{6}$$

For each $\sigma < t < T(\sigma,\phi,g,E(\cdot))$ and $0 < \eta < 1$, we set

$$\begin{aligned}\mathcal{P}^{[t,t+\eta]}_{\phi,g,u}(b) = \Big\{&x : (-\infty, t+\eta] \to X; x\,|_{[t,t+\eta]} \in C([t,t+\eta],X), \\ &\max_{t\leq\tau\leq t+\eta} \|x(\tau) - g(\tau) + g(t) - u(t)\| \leq b, x\,|_{(-\infty,t]} = u\,|_{(-\infty,t]}\Big\}.\end{aligned}$$

Then $\mathcal{P}^{[t,t+\eta]}_{\phi,g,u}(b)$ is a closed convex subset of $\mathcal{P}^{[\sigma,t+\eta]}$. From (H3) and (6) it follows that

$$\max_{\tau\in[\sigma,t+\eta]} \{\|x(\tau)\|, \|x_\tau\|\} \leq b_3, \qquad x \in \mathcal{P}^{[t,t+\eta]}_{\phi,g,u}(b).$$

Here $b_3 = \max\{b + 3b_2, b_2(b + 3b_2 + \|\phi\|_{\mathcal{P}})\}$.

For every $x \in \mathcal{P}_{\phi,g,u}^{[t,t+\eta]}(b)$, define

$$(\mathbf{F}x)(s) = \begin{cases} g(s) - g(t) + u(t) + \displaystyle\int_t^s E(s-\mu) f(\mu, x(\mu), x_\mu)\, d\mu, & t \le s \le t+\eta, \\ u(s), & s \in (-\infty, t]. \end{cases}$$

Using similar arguments as in the proof of Theorem 1, we can deduce that there exists a $0 < \tau(\sigma,\phi,g,E(\cdot)) < 1$, which *is independent of* $t \in [\sigma, T(\sigma,\phi,g,E(\cdot)))$, such that $\mathbf{F}$ has a unique fixed point $x(\cdot)$ in $\mathcal{P}_{\phi,g,u}^{[t,t+\tau(\sigma,\phi,g,E(\cdot))]}(b)$. This $x(t)$ is just the continuous solution of (4) on $[t, t+\tau(\sigma,\phi,g,E(\cdot))]$. Taking t such that $0 < T(\sigma,\phi,g,E(\cdot)) - t < \tau(\sigma,\phi,g,E(\cdot))$, then $t + \tau(\sigma,\phi,g,E(\cdot)) > T(\sigma,\phi,g,E(\cdot))$. This is in contradiction with the definition of $T(\sigma,\phi,g,E(\cdot))$. As a consequence we get $\limsup_{t \uparrow T(\sigma,\phi,g,E(\cdot))} \|u(t)\| = \infty$.

COROLLARY 3. *Let $\sigma \ge 0$, and let $f \in C([\sigma,\infty) \times X \times \mathcal{P}, X)$ satisfy (3) (for $T = \infty$) and*

$$\|f(t,x,\phi)\| \le h_1(t)\|x\| + h_2(t)\|\phi\|_{\mathcal{P}} + h_3(t), \qquad t \in [\sigma,\infty), \quad x \in X, \quad \phi \in \mathcal{P}, \tag{7}$$

where h_i $(i = 1,2,3)$ are locally integrable functions in $[\sigma,\infty)$. Then $T(\sigma,\phi,g,E(\cdot)) = \infty$ for any $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,\infty),X)$ with $g(\sigma) = \phi(0)$, and strongly continuous family $\{E(t)\}_{t\ge 0} \subset \mathbf{L}(X)$.

PROOF. Fix $\sigma \ge 0$, $\phi \in \mathcal{P}$, $g(t) \in C([\sigma,\infty),X)$ with $g(\sigma) = \phi(0)$, and $\{E(t)\}_{t\ge0} \subset \mathbf{L}(X)$ which is strongly continuous. Let $u(t)$ be the corresponding solution to (4) on $[\sigma, T(\sigma,\phi,g,E(\cdot)))$. By (4), (7), and (H3), we obtain for any $t \in [\sigma, T(\sigma,\phi,g,E(\cdot)))$,

$$\begin{aligned} \|u(t)\| \le \|g(t)\| + \int_\sigma^t \|E(t-s)\| \Bigg[& h_1(s)\|u(s)\| + h_2(s) K(s-\sigma) \sup_{\sigma\le\tau\le s} \|u(\tau)\| \\ & + h_2(s) M(s-\sigma)\|\phi\|_{\mathcal{P}} + h_3(s) \Bigg]\, ds. \end{aligned}$$

Therefore, for each $\sigma < T_0 < T(\sigma,\phi,g,E(\cdot)))$ and $t \in [\sigma, T_0)$,

$$\begin{aligned} \sup_{\sigma\le\tau\le t} \|u(\tau)\| \le{} & \sup_{\sigma\le\tau\le T_0} \|g(\tau)\| + \sup_{\sigma\le\tau\le T_0} \|E(\tau)\| \\ & \times \left[\sup_{0\le\tau\le T_0-\sigma} \|M(\tau)\| \|\phi\|_{\mathcal{P}} \int_\sigma^{T_0} h_2(s)\, ds + \int_\sigma^{T_0} h_3(s)\, ds \right] \\ & + \sup_{\sigma\le\tau\le T_0} \|E(\tau)\| \int_\sigma^t [h_1(s) + h_2(s) K(s-\sigma)] \sup_{\sigma\le\tau\le s} \|u(\tau)\|\, ds. \end{aligned}$$

Thus, by Gronwall-Bellman's inequality, for any $\sigma < T_0 < T(\sigma,\phi,g,E(\cdot))$ and $t \in [\sigma, T_0)$,

$$\begin{aligned} \sup_{\sigma\le\tau\le t} \|u(\tau)\| \le{} & \Bigg\{ \sup_{\sigma\le\tau\le T_0} \|g(\tau)\| + \sup_{\sigma\le\tau\le T_0} \|E(\tau)\| \\ & \times \left[\sup_{0\le\tau\le T_0-\sigma} M(\tau) \|\phi\|_{\mathcal{P}} \int_\sigma^{T_0} h_2(s)\, ds + \int_\sigma^{T_0} h_3(s)\, ds \right] \Bigg\} \\ & \times \exp\left\{ \sup_{\sigma\le\tau\le T} \|E(\tau)\| \left[\int_\sigma^{T_0} h_1(s)\, ds + \sup_{0\le\tau\le T_0-\sigma} K(\tau) \int_\sigma^{T_0} h_2(s)\, ds \right] \right\} \\ \le{} & \text{const.} \end{aligned}$$

This, together with the arbitrariness of T_0 and Theorem 2, shows that $T(\sigma,\phi,g,E(\cdot)) = \infty$.

As two consequences of Theorem 1, we obtain the following existence and uniqueness theorems for mild solutions of (1) and (2).

COROLLARY 4. *Let $E \in \mathbf{L}(X)$, and let $\{E(t)\}_{t\geq 0}$ be an E-existence family for A (see [1,13]), that is, $\{E(t)\}_{t\geq 0}$ is a strongly continuous family in $\mathbf{L}(X)$ such that for each $x \in X$, $t \geq 0$,*

$$\int_0^t E(s)x\,ds \in \mathcal{D}(A) \qquad \text{and} \qquad A\left(\int_0^t E(s)x\,ds\right) = E(t)x - Ex.$$

Assume that the zero function is the unique continuous solution of $x(t) = A\int_0^t x(s)\,ds$ $(t \geq 0)$, $E^{-1}f \in C([0,T]\times X\times\mathcal{P}, X)$ and satisfies (3) (for $\sigma = 0$). Then for each $\phi \in \mathcal{P}$ with $\phi(0) \in \mathcal{R}(E)$, there exists a real number $\tau(\phi)$ such that (1) has a unique mild solution $u(t)$ on $[0,\tau(\phi))$ given by $u(t) = E(t)z + \int_0^t E(t-s)E^{-1}f(s,u(s),u_s)\,ds$ $(0 \leq t < \tau(\phi))$, where $z \in X$ and $Ez = \phi(0)$.

REMARK 5. Existence families, introduced by deLaubenfels, generalize classical C_0 semigroups as well as regularized semigroups (cf., e.g., [13,14]). The existence family in Corollary 4 is called a mild existence family in [13].

COROLLARY 6. *Let $[\mathcal{D}(A)]$ be the Banach space $\mathcal{D}(A)$ with the graph norm. Assume that*

(i) *A generates a strongly continuous semigroup on X;*
(ii) *$F(t)([\mathcal{D}(A)]) \subset [\mathcal{D}(A)]$ $(t \in [0,T])$, $AF(\cdot)u(\cdot) \in L^1([0,T],X)$ $(u(\cdot) \in C([0,T],[\mathcal{D}(A)]))$, and $F'(\cdot)u \in C([0,T],X)$ $(u \in X)$;*
(iii) *$f \in C([0,T]\times X\times\mathcal{P}, X)$ and satisfies (3) (for $\sigma = 0$).*

Then for any $\phi \in \mathcal{P}$, there exists a real number $\tau(\phi)$ such that (2) has a unique mild solution $u(t)$ on $[0,\tau(\phi))$ given by $u(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s,u(s),u_s)\,ds$ $(0 \leq t < \tau(\phi))$, where $\{R(t)\}_{0\leq t\leq T}$ is a resolvent operator of (2) with $f \equiv 0$ (see [11]), that is, $\{R(t)\}_{0\leq t\leq T} \subset \mathbf{L}(E)$ and satisfies the following properties:

(i) *$R(0) = I$ (the identity operator on X) and $R(t)$ is strongly continuous on $[0,T]$;*
(ii) *for all $u \in [\mathcal{D}(A)]$, $R(\cdot)u \in C^1([0,T],X) \cap C([0,T],[\mathcal{D}(A)])$, and for all $0 \leq t \leq T$,*

$$R'(t)u = A\left[R(t)u + \int_0^t F(t-s)R(s)u\,ds\right] = R(t)Au + \int_0^t R(t-s)AF(s)u\,ds.$$

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